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Equation of a Sphere

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The segment $\overline{OQ}$ is created b the projection of P on $\overline{OQ}$ and is also the bisection of the rectangle described by $\overline{O, y-axis, x-axis, Q}$. $\overline{Q, x-axis}$ is a perpendicular projection on the x-axis and $\overline{Q, y-axis}$ is a perpendicular projection on the y-axis.

In the triangle between $\overline{OQ}$ and the x-axis:

$$\cos(\beta) = \frac{x}{\overline{OQ}}$$

In the triangle between $\overline{OQ}$ and the y-axis:

$$\sin(\beta) = \frac{y}{\overline{OQ}}$$

In the triangle given by $\widehat{OPQ}$:

$$\cos\alpha = \frac{\overline{OQ}}{\overline{OP}} = \frac{OQ}{r}$$

Now, we take the last equation:

$$\cos\alpha = \frac{\overline{OQ}}{r}$$

$\text{and take out } \overline{OQ}$:

$$OQ = r * \cos\alpha$$

and substitute $\overline{OQ}$ in the first two equations:

$$\begin{aligned}\cos\beta &= \frac{x}{r * \cos\alpha} \\ \sin\beta &= \frac{y}{r * \cos\alpha}\end{aligned}$$

we extract x and y since it is what we are looking for:

$$\begin{aligned}x &= r * \cos\alpha * \cos\beta \\ y &= r * \cos\alpha * \sin\beta\end{aligned}$$

we also need z, so we go back to the triangle $\widehat{OPQ}$:

$$\sin\alpha = \frac{\overline{PQ}}{\overline{OP}} = \frac{z}{r}$$

so we extract z:

$$z = r * \sin \alpha$$

Now, concluding, we obtain:

$$\begin{aligned} x &= r * \cos \alpha * \cos \beta \\ y &= r * \cos \alpha * \sin \beta \\ z &= r * \sin \alpha \end{aligned}$$

which is the parametric form of the equation of a sphere in space. If you follow what we did, we really did the same thing that we did for a sphere, but we just did it several times in order to obtain z as well.

Volume of a Sphere

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A sphere can be seen as infinitely many disks placed on top of each other, each representing the cross-section of the sphere at a certain point along the z axis.

This means that we can represent the difference in volume as a summation of all the surfaces of the circles (each having the area $A_{\bullet} = \pi * r^2$) aligned along the z axis and write that:

$$V \approx \sum A_{\bullet}^{\infty} * \delta z$$

if we apply this to the whole range on the z axis while sweeping the dimension of the circles starting at the south and up to the north pole, in both cases where $y = 0$. This makes the radius range from $-r$ at the south pole to r at the north pole and the equation becomes:

$$V = \int_{-r}^{r} \pi * y^2 * dz$$

At any given z, a right-angle triangle is formed by (for example) $\widehat{O, P, Q}$ in which we can apply Pythagoras. This translates to an universal right-angle triangle in point Q so that at any value of z we can map:

$$\begin{aligned} \overline{OQ} &\mapsto z \\ \overline{OP} &\mapsto r \\ \overline{QP} &\mapsto y \end{aligned}$$

and leading to Pythagoras:

$$y^2 = r^2 - z^2$$

Now we can replace y^2 in the integral equation and obtain:

$$V = \int_{-r}^{r} \pi * (r^2 - z^2) * dz$$

we can take out the constant π and rewrite the integral as:

$$V = \pi * \int_{-r}^{r} (r^2 - z^2) * dz$$

and expand using definite integral rules to:

$$\begin{aligned} V &= \pi * [\int_{-r}^{0} r^2 dz - \int_{0}^{r} z^2 * dz] \\ &= \pi * [r^2 z \Big|_{-r}^{0} - \frac{1}{3} z^3 \Big|_{0}^{-r}] \\ &= \pi * \{(r^2 * 0) - (r^2 * -r) - [\frac{1}{3}(-r)^3 - \frac{1}{3}(0)^3]\} \\ &= \pi * (r^3 + \frac{1}{3} * r^3) \\ &= \pi * (\frac{3 * r^3 + r^3}{3}) \\ &= \pi * (\frac{4 * r^3}{3}) \end{aligned}$$

and rearranging the terms nicely, we obtain the volume as:

$$V = \frac{4}{3} \pi * r^3$$

Area of a Sphere

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The total volume inside a sphere of radius r can be thought of as the summation of the surface area $A(r)$ of an infinite number of spherical shells of infinitesimal thickness concentrically stacked inside one another from radius 0 to radius r.

At any given radius r the incremental volume δV equals the product of the surface area at radius $r(A(r))$ and the thickness of the shell δr.

Thus, we can write that:

$$V \approx \sum A(r) \delta r$$

as δr approaches zero, we can integrate from 0 to r (all the internal shells, from radius zero to the radius of the big sphere).

$$V = \int_0^r A(r)dr$$

We have already derived the volume, so we can now substitute the volume into the equation and rewrite:

$$\frac{4}{3}\pi * r^3 = \int_0^r A(r)dr$$

Now we differentiate both sides of the equation with respect to r in order to get rid of the integral and derive the area A as a function of the radius r:

$$\frac{d}{dr}\left[\frac{4}{3}\pi * r^3\right] = \frac{d}{dr}\left[\int_0^r A(r)dr\right]$$

Differentiating on the left side, we obtain:

$$\frac{4}{3}\pi * 3 * r^2 = \frac{d}{dr}\left[\int_0^r A(r)dr\right]$$

while differentiating on the right side, the defined integral from 0 to r does not matter, we just use the differentiation integral rule:

$$\frac{d}{d\mathbf{x}}\left[\int \mathbf{x}d\mathbf{x}\right] = \mathbf{x}$$

and obtain:

$$\frac{4}{3}\pi * 3 * r^2 = A(r)$$

Finally, simplifying on the left side:

$$4\pi r^2 = A(r)$$

and turning the equality around so it looks like a forumula, yields the area of the sphere:

$$A(r) = 4\pi r^2$$

Since we already know that the area is a function of r, we can abbreviate this to:

$$A = 4\pi r^2$$

and obtain the final area of the large sphere.